Paraconsistent and Paracomplete Set Theory

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1. Motivation

Classical, paracomplete, and paraconsistent logic

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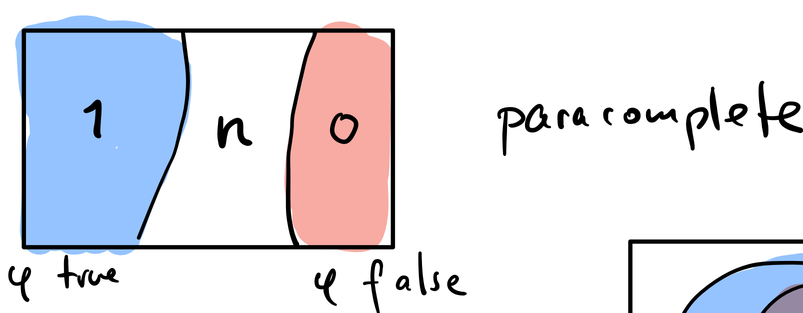
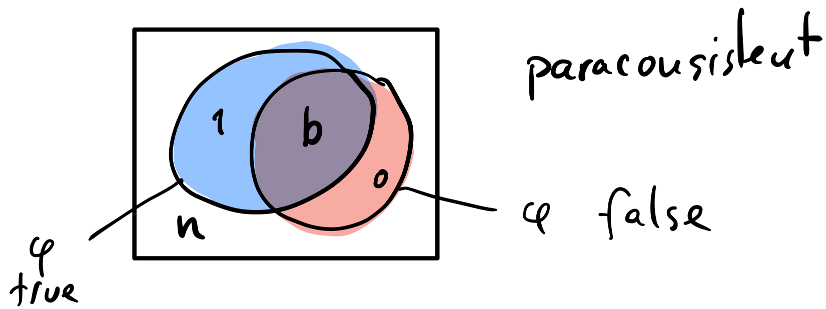
- Paracomplete and paraconsistent: φ can be **true**, **false**, **neither**, or **both**.





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Why?

- Usually paraconsistent set theory is motivated by a desire to satisfy **full comprehension** and avoid Russell's Paradox.
- Our approach is different: we do not consider Russell's Paradox as something that must be avoided.
- Rather, the goal is to provide a mathematically coherent foundation for paraconsistent and paracomplete sets with a clear **model theory**.
- Carefully looking at the axioms and translating them correctly—seems to overcome several obstacles faced by previous authors.

Definition

- Usual syntax of FOL (negation denoted by $\sim$).
- A **True/False-model** $\mathcal{M}$ consist of a domain, and for every relation symbol R (including $=$)
  - a "positive" interpretation $(R^{\mathcal{M}})^{+}$ and
  - a "negative" interpretation $(R^{\mathcal{M}})^{-}$.

2. The logic BS4

Definition

- $\mathcal{M} \models^T R(t_1, \dots, t_n)[\dots] \iff R^+(t_1^{(\mathcal{M},\dots)}, \dots, t_n^{(\mathcal{M},\dots)})$ holds.
  $\mathcal{M} \models^F R(t_1, \dots, t_n)[\dots] \iff R^-(t_1^{(\mathcal{M},\dots)}, \dots, t_n^{(\mathcal{M},\dots)})$ holds.
- $\mathcal{M} \models^T \sim\varphi \iff \mathcal{M} \models^F \varphi$.
  $\mathcal{M} \models^F \sim\varphi \iff \mathcal{M} \models^T \varphi$.
- $\mathcal{M} \models^T \varphi \wedge \psi \iff \mathcal{M} \models^T \varphi$ and $\mathcal{M} \models^T \psi$.
  $\mathcal{M} \models^F \varphi \wedge \psi \iff \mathcal{M} \models^F \varphi$ or $\mathcal{M} \models^F \psi$.
- $\mathcal{M} \models^T \varphi \to \psi \iff$ if $\mathcal{M} \models^T \varphi$ then $\mathcal{M} \models^T \psi$.
  $\mathcal{M} \models^F \varphi \to \psi \iff \mathcal{M} \models^T \varphi$ and $\mathcal{M} \models^F \psi$.
- $\mathcal{M} \models^T \exists x \varphi(x) \iff \mathcal{M} \models^T \varphi[a]$ for some $a \in M$.
  $\mathcal{M} \models^F \exists x \varphi(x) \iff \mathcal{M} \models^F \varphi[a]$ for all $a \in M$.
- $\mathcal{M} \models^T \perp \iff$ never.
  $\mathcal{M} \models^F \perp \iff$ always.

2. The logic BS4

Definition

- $\varphi \vdash_{\mathsf{BS4}} \psi$ means that for all True/False-models $\mathcal{M}$, if $\mathcal{M} \models^T \varphi$ then $\mathcal{M} \models^T \psi$.
- It is not hard to formalize a sound and complete proof calculus for BS4 (but we will skip this).

2. The logic BS4

Truth Value

$$[\![\varphi]\!]^{\mathcal{M}} := \begin{cases} 1 & \text{if } \mathcal{M} \models^T \varphi \text{ and } \mathcal{M} \not\models^F \varphi \\ \mathfrak{b} & \text{if } \mathcal{M} \models^T \varphi \text{ and } \mathcal{M} \models^F \varphi \\ \mathfrak{n} & \text{if } \mathcal{M} \not\models^T \varphi \text{ and } \mathcal{M} \not\models^F \varphi \\ 0 & \text{if } \mathcal{M} \not\models^T \varphi \text{ and } \mathcal{M} \models^F \varphi \end{cases}$$

2. The logic BS4

	$\sim$
1	0
$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{n}$	$\mathfrak{n}$
0	1

$\wedge$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	0	0
$\mathfrak{n}$	$\mathfrak{n}$	0	$\mathfrak{n}$	0
0	0	0	0	0

$\vee$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	1	1	1
$\mathfrak{b}$	1	$\mathfrak{b}$	1	$\mathfrak{b}$
$\mathfrak{n}$	1	1	$\mathfrak{n}$	$\mathfrak{n}$
0	1	$\mathfrak{b}$	$\mathfrak{n}$	0

$\to$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	1	1	1	1
0	1	1	1	1

$\leftrightarrow$	1	$\mathfrak{b}$	$\mathfrak{n}$	0
1	1	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{n}$	0
$\mathfrak{n}$	$\mathfrak{n}$	$\mathfrak{n}$	1	1
0	0	0	1	1

	$\perp$
	0

2. The logic BS4

Is $\perp$ against the "spirit" of paraconsistent logic?

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Is $\bot$ against the "spirit" of paraconsistent logic?

- If we only have finitely many relation symbols $R_1, \ldots, R_n$ then

$$\bot \quad \equiv \quad \forall x \forall y (x = y \wedge \sim (x = y)) \wedge \forall \bar{x} (R_1(\bar{x}) \wedge \sim R_1(\bar{x})) \wedge \ldots \wedge \forall \bar{x} (R_n(\bar{x}) \wedge \sim R_n(\bar{x}))$$

Only the "trivial model" can satisfy $\bot$.

BS4 can talk about classical concepts

- We can define the **classical negation**:

$$\neg\varphi \quad \equiv \quad \varphi \to \bot$$

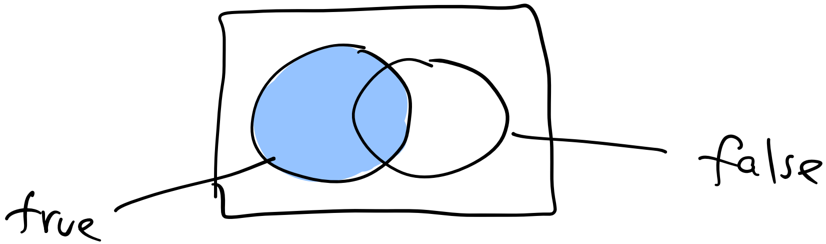
- We can define **presence of truth**:

$$!\varphi \quad \equiv \quad \sim\neg\varphi$$

- We can define **absence of falsity**:

$$?\varphi \quad \equiv \quad \neg\sim\varphi$$

2. The logic BS4



φ	$\sim\varphi$	$\neg\varphi$	$!\varphi$	$?\varphi$
1	0	0	1	1
$\mathfrak{b}$	b	0	1	0
$\mathfrak{n}$	n	1	0	1
0	1	1	0	0

presence of truth (the $!\varphi$ column)

absence of falsity (the $?\varphi$ column)

3. Paraconsistent and Paracomplete set theory

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  - Extension: those y for which $y \in x$ is **true**.
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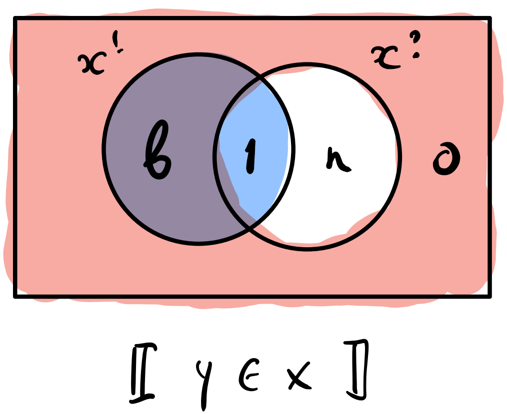
- In a paraconsistent and paracomplete logic:
  - Extension: those y for which $y \in x$ is **true**.
  - Anti-extension: those y for which $y \in x$ is **false**.
- Problem: anti-extensions are **proper classes**.
- Instead, we will talk about the **complement** of the anti-extension:
  - ?-extension: those y for which $y \in x$ is **not false**.

Extension and ?-extension

$$x^{!} := \{y \mid \; !(y \in x)\}$$

$$x^{?} := \{y \mid \; ?(y \in x)\}$$



3. Paraconsistent and Paracomplete set theory

Consistent, complete and classical sets

- x is **consistent** if $x^! \subseteq x^?$
- x is **complete** if $x^? \subseteq x^!$
- x is **classical** if $x^! = x^?$

These statements can be expressed in BS4

3. Paraconsistent and Paracomplete set theory

*)

The PZFC Axioms

1. Extensionality: $\forall x \forall y (x = y \Leftrightarrow \forall z (z \in x \Leftrightarrow z \in y)$
2. Comprehension: $\forall u \exists x \forall y \, (y \in x \Leftrightarrow y \in u \wedge \varphi(y))$
3. Classical superset: $\forall x \, \exists C \, (x \subseteq C \;\wedge\; C$ is classical)
4. Replacement: (φ is classical $\wedge \forall x \exists y (\varphi(x,y) \wedge \forall z (\varphi(x,z) \rightarrow !(y = z))) \rightarrow$
   $\forall x \exists y \forall z (z \in y \Leftrightarrow \exists w (w \in x \wedge \varphi(w,z)))$
5. Pairing: $\forall u \forall v \exists x \forall y (y \in x \;\Leftrightarrow\; (!(y = u) \vee !(y = v)))$
6. Power Set: $\forall u \exists x \forall z (z \in x \;\Leftrightarrow !(z \subseteq u))$
7. Union: $\forall u \exists x \forall y (y \in x \;\Leftrightarrow\; \exists z \, (y \in z \wedge z \in u)$
8. Infinity: $\exists x (\varnothing \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$
9. Foundation: $\forall x (\forall y \in (x^! \cup x^?) \varphi(y) \rightarrow \varphi(x)) \;\rightarrow\; \forall x \varphi(x)$
10. Choice: $\forall u (\forall x \in u \, \exists y \in x)) \;\rightarrow\; \exists f$ function s.t. $\forall x \in u (f(x) \in x)$

*) $(\varphi \Rightarrow \psi)$ means $(\varphi \to \psi) \wedge (\sim\psi \to \sim\varphi)$

4. The anti-classicality axiom.

- None of the axioms of PZFC guarantee the existence of non-classical sets. In fact

$$\mathsf{PZFC} + \forall x (x^! = x^?) \equiv \mathsf{ZFC}$$

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- We want to set up a system that can prove the existence of non-classical sets.
- But which ones exactly?
  - Conservative: there exists **at least one** inconsistent and incomplete set.
  - Maximizing: for **any** pair of classical sets u, v there is a set x whose extension is u and $?$-extension is v.

4. The anti-classicality axiom.

Theorem (PZFC)

Suppose there exists an incomplete set and an inconsistent set. Then for **any** classical sets u, v, there exists a set x such that

$$x^{!} = u \quad \text{and} \quad x^{?} = v$$

4. The anti-classicality axiom.

Definition

We add the Anti-Classicality Axiom to our system:

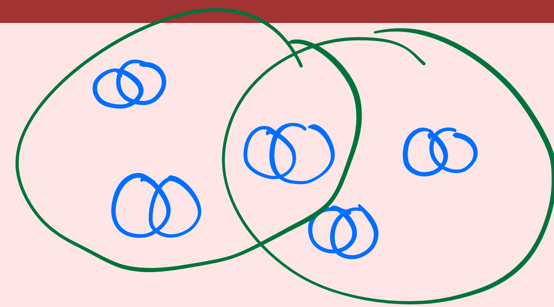
$$\textbf{ACLA:} \quad \exists x (x \text{ is inconsistent}) \wedge \exists x (x \text{ is incomplete}).$$

and extend the theory:

$$\text{BZFC} \quad \equiv \quad \text{PZFC} + \text{ACLA}.$$

5. Model of BZFC and bi-interpretability.

Definition (ZFC)



- $W_0 := \varnothing$
- $W_{\alpha+1} := \mathscr{P}(W_\alpha) \times \mathscr{P}(W_\alpha)$
- $W_\lambda := \bigcup_{\alpha<\lambda} W_\alpha$
- $\mathbb{W} := \bigcup_\alpha W_\alpha$

For all $(a, b), (c, d) \in \mathbb{W}$ define

- $(a, b) \in^+ (c, d)$ iff $(a, b) \in c$
- $(a, b) \in^- (c, d)$ iff $(a, b) \notin d$
- $(a, b) =^+ (c, d)$ iff $(a, b) = (c, d)$
- $(a, b) =^- (c, d)$ iff $\exists z \in a \setminus d$ or $\exists z \in c \setminus b$.

5. Model of BZFC and bi-interpretability.

Theorem (ZFC)

$(\mathbb{W}, \in^+, \in^-, =^+, =^-) \models \text{BZFC}.$

Corollary

If ZFC is consistent, then BZFC is non-trivial (i.e., BZFC $\not\vdash \bot$).

5. Model of BZFC and bi-interpretability.

Definition (BZFC)

- $\mathrm{HCL}_0 := \varnothing$
- $\mathrm{HCL}_{\alpha+1} := \{X \subseteq \mathrm{HCL}_\alpha \mid X \text{ is classical}\}$
- $\mathrm{HCL}_\lambda := \bigcup_{\alpha<\lambda} W_\alpha$
- $\mathbb{HCL} := \bigcup_\alpha \mathrm{HCL}_\alpha$

$\mathbb{HCL}$ is the class of **hereditarily classical sets**.

5. Model of BZFC and bi-interpretability.

Theorem (BZFC)

$\mathbb{HCL} \models \mathsf{ZFC}$.

Corollary

If BZFC is non-trivial, then ZFC is consistent.

5. Model of BZFC and bi-interpretability.

In fact, we can be more precise. Let V denote the universe of ZFC and W the universe of BZFC.

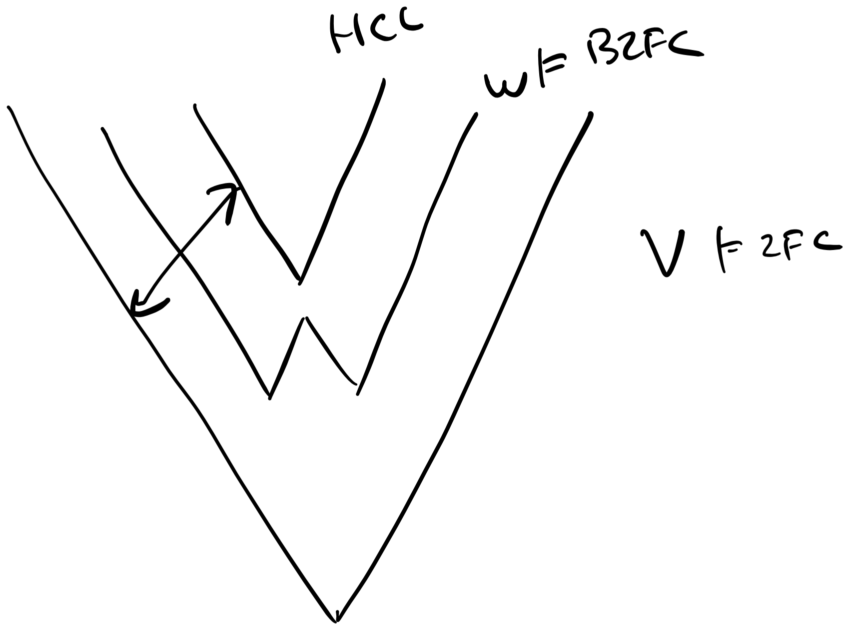
Theorem (ZFC)

$$\mathbb{HCL}^{\mathbb{W}} \cong V$$

Theorem (BZFC)

$$\mathbb{W}^{\mathbb{HCL}} \cong W$$

5. Model of BZFC and bi-interpretability.



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Corollary

- ZFC $\vdash \varphi$ iff BZFC $\vdash (\mathbb{HCL} \models \varphi)$
- BZFC $\vdash \varphi$ iff ZFC $\vdash (\mathbb{W} \models \varphi)$

5. Model of BZFC and bi-interpretability.

If we want a paracomplete and paraconsistent set theory:

> BZFC describes a rich universe consisting of classical and non-classical sets. ZFC can then be viewed as the theory of $\mathbb{HCL}$, and all of classical mathematics as taking place within $\mathbb{HCL}$. Whenever we encounters a phenomenon that is better described by paracomplete or paraconsistent sets, we can switch to BZFC and take full advantage of the anti-classicality axiom.

If we want to preserve a classical meta-theory:

> BZFC can be viewed as the theory of the True/False-model $\mathbb{W}$, and all of paraconsistent and paracomplete set theory as taking place within $\mathbb{W}$ (provably in ZFC).

Thank you!

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